circulant matrix

$$\text{circcorr}(\vec{x},\vec{y}) = \begin{bmatrix} - \vec{y}(0)^T - \\ - \vec{y}(1)^T - \\ \vdots \\ - \vec{y}(N-1)^T - \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix}$$

$$C_{\vec{y}} = \begin{bmatrix} y[0] & y[1] & \cdots & y[N-2] & y[N-1] \\ y[N-1] & y[0] & \cdots & y[N-3] & y[N-2] \\ \vdots & \vdots & & \vdots & \vdots \\ y[2] & y[3] & \cdots & y[0] & y[1] \\ y[1] & y[2] & \cdots & y[N-1] & y[0] \end{bmatrix}$$

ⓒ ~~[illegible]~~ Now it's 2D, not 1D. Given $\vec{r}$ from Part ⓑ write an expression for all possible solns in terms of x_{mic}, y_{mic}.
note: $x_A = -10, y_A = 0, x_B = 10, y_B = 0$

Beacon A: $\sqrt{(x_{mic} - x_A)^2 + (y_{mic} - y_A)^2} = d_A$

Beacon B: $\sqrt{(x_{mic} - x_B)^2 + (y_{mic} - y_B)^2} = d_B$

from ⓒ ↝ $d_B = d_A + 2$

$\therefore \sqrt{(x_{mic} - x_B)^2 + (y_{mic} - y_B)^2} = d_A + 2$

Beacon B eqn - Beacon A eqn = 2

$-\sqrt{(x_{mic} - x_A)^2 + (y_{mic} - y_A)^2} + \sqrt{(x_{mic} - x_B)^2 + (y_{mic} - y_B)^2} = 2$

$\text{corr}_{\vec{x}}(y)[k] = \text{corr}_{\vec{y}}(\vec{x})[-k]$

ⓐ If 2 beacons transmit signals at the same time and the received signal is equal distance btwn them, the received signal is a sum of the two beacon signals.

$\vec{A} = [1\ 0\ 1\ 0\ 0\ 0\ 0]$ $\vec{B} = [1\ 0\ 0\ 1\ 0\ 0\ 0]$

$d_A = |-10| = 10$ $d_B = 10$

$\vec{r} = \vec{A} + \vec{B} = [2\ 0\ 1\ 1\ 0\ 0\ 0]$

ⓑ What the smallest time delay btwn when the signals arrive? (TDOA)

$\text{corr}_{\vec{r}}(\vec{A}) = [2\ 1\ 0\ 2\ 0\ 3\ 0]$ → peak @ $n=5$ or $n=-2$

$\text{corr}_{\vec{r}}(\vec{B}) = [3\ 0\ 1\ 2\ 1\ 1\ 0]$ → peak @ $n=0$

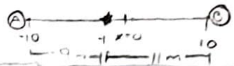
↳ tells us that $\vec{A}$ is lagging by 5 timesteps or leading by 2

↳ smallest delay = leading by 2

ⓒ Based on ⓑ where is the mic located in the 1D system?

$d_B - d_A = (1\ m/s)(2\ s) = 2\ m$

$2m = 11m - 9m$



mic is at $x = -1$

ⓓ What are all integer soln of where the mic is located in the 1D system?

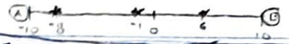
length of signal $N = 7$

period of the signal = 7

$7/2 = 3.5$ m in each direction

→ but only want integer solns

↳ more 1 in each dir.



say we have

where's the prob?

- C is colinear with A & B (putting them in the same line won't give us a unique soln

Projection onto a subspace

- If columns of A are orthogonal, $\text{Proj}_A \vec{b} = \sum \text{proj}_{\vec{a}_i}(\vec{b})$ where $\vec{a}_i$ are the columns of A.
- If columns A not orthogonal, use least squares:

$\hat{\vec{b}} = A(A^TA)^{-1}A^T\vec{b}$

~~To minim~~ Least Sq.

To minimize the error $\vec{e} = \|A\vec{x} - \vec{b}\|$ we have $\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$

A^TA is invertible if A has LI columns (can only use least sq. when A has LI columns)

Trilateration

- n variables
  - n eqn if linear
  - n+1 if nonlinear
- in space (n-dim)
  - n+1 for circle/sphere
  - n+2 if eqn unknown

Units

$\vec{\cos t} = \vec{c}$

- current: $I = \frac{Q}{t}$ = charge/time
- voltage: $V = J/Q$
- cap: $C = \frac{Q}{V}$
- $\Omega = V/I$
- power = J/s

Metric Conversions

peta (p): E15
tera (T): E12
giga (G): E9
mega (M): E6
kilo (k): E3
hecto (h): E2
deca (da): E1
base (b): E0
deci (d): E-1
centi (c): E-2
milli (m): E-3
micro (μ): E-6
nano (n): E-9
pico (p): E-12
femto (f): E-15

Exponent Rules

$\frac{x^m}{x^n} = x^{m-n}$

$x^m y^n = x^{m+n}$

$(xy)^n = x^n y^n$

$(x^m)^n = x^{mn}$

KCL $I_{in} = I_{out}$

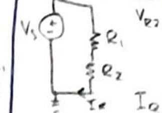
KVL $\sum_{loop} V_k = 0$; + → − add, − → + subtract

Ohm $V = IR$

Passive Devices Resistors & Caps

	resistors	capacitors
Construction	$R=\rho\frac{L}{A}$	$C=K\epsilon_0\frac{A}{d}=\epsilon\frac{A}{d}$
IV-relation	$V=IR$	$Q=CV_c$, $i=C\frac{dV}{dt}$
series eq.	$R_1+R_2+\dots$	$C_1 \parallel C_2$ $\left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}$
parallel eq.	$R_1 \parallel R_2$	C_1+C_2
energy store	none	$E=\frac{1}{2}CV^2$

Voltage Divider (series)

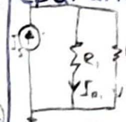


$V_{R2} = \frac{V_s R_2}{R_1+R_2}$

$I_R = \frac{V_s}{R_1+R_2}$

$V_{R1} = \frac{V_s R_1}{R_1+R_2}$

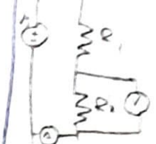
Current Divider (parallel)



$I_{R1} = I_s \frac{R_2}{R_1+R_2}$ $V_{R1} = V_{R2}$

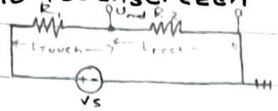
Source is the most ... located

Measurement



Power $P = VI = \frac{V^2}{R} = I^2R$

1D Touchscreen



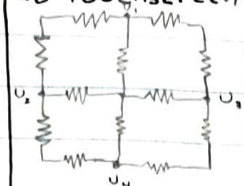
$R_1 = \frac{\rho L_{touch}}{A}$ $R_2 = \frac{\rho L_{rest}}{A}$

$U_{mid} = V_s \frac{\frac{\rho L_{rest}}{A}}{\frac{\rho L_{rest}}{A} + \frac{\rho L_{touch}}{A}}$

$U_{mid} = V_s \frac{L_{rest}}{L_{rest}+L_{touch}}$

$U_{mid} = V_s \frac{L_{rest}}{L_{TOTAL}}$

2D Touchscreen



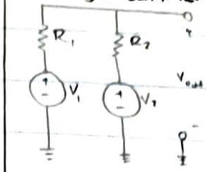
Powering $U_1 \to U_4$: measure x position

$V_{out} = V_s \frac{L_{touch,vertical}}{L}$

Powering $U_2 \to U_3$: measure y position

$V_{out} = V_s \frac{L_{touch,horizontal}}{L}$

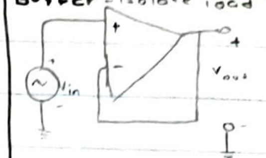
Voltage Summer



$V_{out} = \frac{V_1 R_2}{R_1+R_2} + \frac{V_2 R_1}{R_1+R_2}$

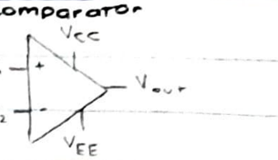
- negative gain
- does NOT have ∞ input resistance
- will load input signal
- recommended to use after a buffer or if you know your input/source has no source resistance

Buffer - isolate load



$V_{out} = V_{in}$

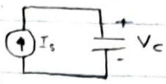
Comparator



$V_{out} = \begin{cases} V_{CC} & \text{if } V_1 > V_2 \\ V_{EE} & \text{if } V_1 < V_2 \end{cases}$

Capacitor

"create a voltage which increases/decreases linearly with respect to time" or a voltage ramp with a given slope.



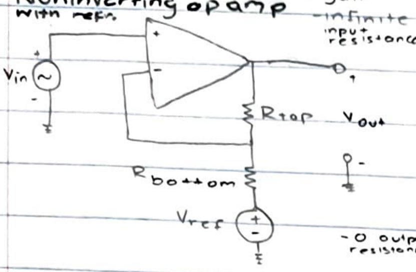
$I_s = C\frac{dV_c}{dt} + V_c\frac{dC}{dt}$

$V_c(t) = \frac{I_s}{C}(t-t_0) + V_c(t_0)$

Noninverting op amp with ref.

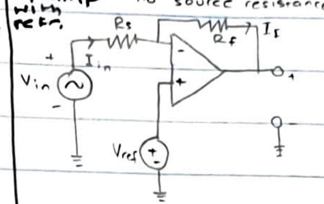
positive gain

- infinite input resistance
- 0 output resistance



$V_{out} = V_{in}\left(1 + \frac{R_{top}}{R_{bottom}}\right) - V_{ref}\left(\frac{R_{top}}{R_{bottom}}\right)$

Inverting opamp with ref.

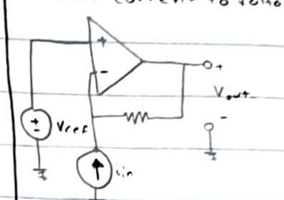


$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right) + V_{ref}\left(\frac{R_f}{R_s} + 1\right)$

$I_{in} = I_{out}$

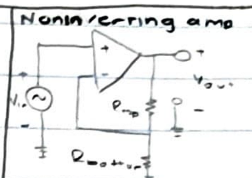
Transresistance amp

convert current to voltage



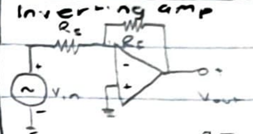
$V_{out} = i_{in}(-R) + V_{ref}$

Noninverting amp



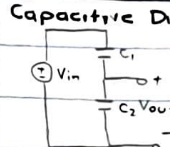
$V_{out} = V_{in}\left(1 + \frac{R_{top}}{R_{bottom}}\right)$

Inverting amp



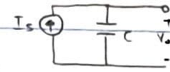
$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right)$

Capacitive Divider:



$V_{out} = \frac{C_1 V_{in}}{C_1 + C_2}$

Charging a capacitor:

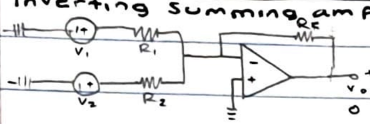


$V_{out} = \frac{I_s t}{C}$

Ideal op amp

$A \to \infty$ $U^+ - U^- = 0$ $U^+ = U^-$

Inverting summing amplifier



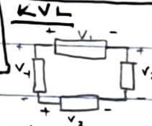
$V_{out} = -R_f\left(\frac{V_{in1}}{R_1} + \frac{V_{in2}}{R_2}\right)$ inner product

Op amps and golden rules

① $I^+ = I^- = 0$

② (NF only) $V^+ = V^-$ (note $V^+ \neq 0 \neq V^-$)

KVL



$-V_1 - V_2 + V_3 + V_4 = 0$

Cap Derivation

$I_s = C\frac{dV(t)}{dt}$

$\int_a^b I_s\,dt = \int_a^b C\frac{dV(t)}{dt}dt$

$I_s t = C(V(t) - V(0))$

$V(t) = \frac{I_s t}{C} + V(0)$

Note: to use this, I_s must be const → choose intervals where it's

Capacitative TS

More power

- negative power dissipated $P<0$
- positive power dissipated $P>0$
- negative power generated $P>0$
- positive power generated $P<0$

Note: 6W generated by V_s <=> −6W dissapated by V_s

Have 2 diff sized cap plates?

measure the smaller plate for the capacitance!

Capacitor Principles

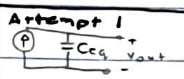
- charge at floating nodes is conserved
- if sum of charges on a floating node = 0, the sum of their charges at steady state will equal zero.
- the charge of a capacitor ⇒ $+Q = +CV$
- if 2 caps are initially uncharged, then connected in series, the charges on both caps are equal at steady state
- voltage across capacitors in parallel is equal at steady state.
- in steady state, DC capacitors act as open circuits (no current flows through them)

Derivation

$I = C\frac{dV_c}{dt}$, $I\,dt = C\,dV_c$, $\int_0^t I\,dt = \int_0^{V_c} C\,dV_c$,

$It = C V_c(t) - V_c(0)$ $V_c(t) = \frac{I}{C}t + V_c(0)$

Attempt 1

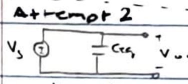


$C_{eq} = C_0 + C_\Delta$

$C_\Delta = \left(\frac{1}{C_{E1F}} + \frac{1}{C_{E2F}}\right)^{-1}$

$I_s = C\frac{dV_c}{dt}$ $V_c = \frac{I_s t}{C_{eq}}$

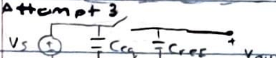
Attempt 2



V_{out} (touch) $= V_s$

V_{out} (no touch) $= V_s$

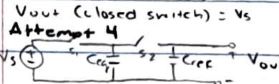
Attempt 3



V_{out} (open switch) $= 0$

V_{out} (closed switch) $= V_s$

Attempt 4

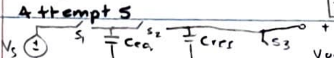


Φ1: close S_1 to charge C_{eq} up, S_2 open

Φ2: open S_1, close S_2

Prob: Potential charge on C_{ref}

Attempt 5



Φ1: S_3 & S_1 on, S_2 off (discharge anything off C_{ref})

$V_{cref} = 0$ $V_{ceq} = V_s$

$Q_{ref} = CV = 0$ $Q_{ceq} = C_{eq}V_s$

Φ2: S_3 & S_1 off, S_2 on

$V_{ceq} = V_{out}$ $V_{cref} = V_{out}$

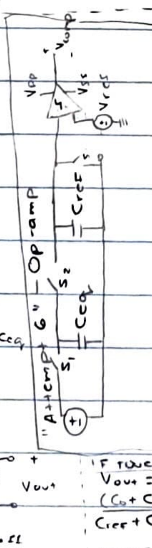
$Q_{Ceq} = C_{eq}V_{out}$ $Q_{cref} = C_{ref}V_{out}$

Total Q @ phase 2: $C_{eq}V_{out} + C_{ref}V_{out}$

If touch: $V_{out} = \frac{(C_0 + C_\Delta)V_s}{C_{ref} + C_0 + C_\Delta}$

If no touch: $V_{out} = \frac{C_0 V_s}{C_{ref} + C_0}$

"Attempt 6" — Op-amp



Charge Sharing

$Q^{\Phi 1} = Q^{\Phi 2}$

① Label all voltages across capacitors

② Draw the equivalent circuit in each phase

③ Identify all floating nodes in your circuit during phase 2. For each node U_i:

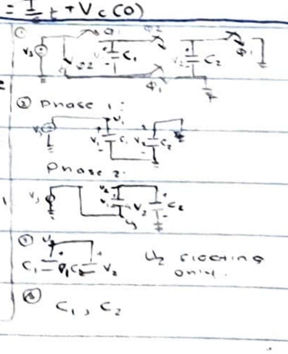
ⓐ identify all cap plates attached to the node during phase 2

ⓑ calculate the charge on each of the plates in the steady state of phase 1

ⓒ calculate the charge on each plate during phase 2's steady state.

ⓓ set $Q^{\Phi 1} = Q^{\Phi 2}$. Solve for the unknown node voltage.

ⓔ Repeat.



① Phase 1

Phase 2

③ U_2 floating only

④ C_1, C_2

ⓑ $Q^{\Phi 1}_{U2} = C_1V_1^{\Phi 1} + C_2V_2^{\Phi 1} = (V_s - 0)C_1 + 0 = V_sC_1$

ⓒ $Q^{\Phi 2}_{U2} = C_1V_1^{\Phi 2} + C_2V_2^{\Phi 2} = (U_2 - U_3)C_1 + U_2C_2 = (U_2 - V_s)C_1 + U_2C_1 = U_2(C_1 + C_2) - V_sC_1$

ⓓ $Q^{\Phi 1}_{U2} = Q^{\Phi 2}_{U2} \Rightarrow 2V_sC_1 = U_2(C_1 + C_2)$ $U_2 = \frac{2V_sC_1}{C_1 + C_2}$

Checking for negative feedback

① Zero out all independent sources, other than the power supply

---

$R = \rho\frac{L}{A}$ (material's length; material's cross section)

$1\,k\Omega = 1000\,\Omega$

$= 1{,}000{,}000\,m\Omega$

$1\,mA = 0.001\,A$

$= 0.000001\,kA$

Power → W

Energy → J

voltage drop: voltage across the base minus the tip $(V_b - V_a)$

$P = IV$ $P = -iV$

consuming power – entering higher node

generating power – exiting higher

resistors consume power; caps consume when charging, generate when discharging

Basic and Free Variables

$x_1\ x_2\ x_3\ x_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Basic vars
Free vars
O Pivots
≥1 free var → ∞ soln
0 free var → 1 soln

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$

$$\left[\begin{array}{ccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & b_1 \\ 0 & 0 & 1 & \# & 5 & 6 & 7 & b_2 \\ 0 & 0 & 0 & 1 & 5 & 6 & 7 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & b_4 \end{array}\right]$$

Matrix Multiplication

for each row of A, multiply and sum for each col of B

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

Matrix Vector Mult

$A_{m\times n}$, $\vec{x}_{n\times 1}$

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Linear Independence

1. $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ LD if $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}$
2. $\vec{v}_i = \sum_{j\neq i} \alpha_j \vec{v}_j$

2 Definitions of Linear ~~[illegible]~~ Dependence

Span

$\text{span}\{v_1, \ldots, v_n\} = \left\{\sum_{i=1}^{n} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{R}\right\}$

- set of all linear combos of $\{v_1, \ldots, v_n\}$
- span of a set of vectors is a subspace
- span(A) = range(A)
- span(A) = columnspace(A)

Is $\vec{v}$ in the span $\{[\vec{a}], [\vec{b}], [\vec{c}]\}$?

To solve this, aug matrix & GE

$\left[\begin{array}{ccc|c} \vec{a} & \vec{b} & \vec{c} & \vec{v} \end{array}\right]$ → if no soln, $\vec{v}$ not in the span

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ Lin ind?

need to find nullspace. If trivial, LI. If nontrivial, LD

$\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{array}\right]$

state-transition matrix

$$A = \begin{bmatrix} P_{1\to1} & P_{2\to1} & \cdots & P_{n\to1} \\ P_{1\to2} & P_{2\to2} & \cdots & P_{n\to2} \\ \vdots & \vdots & & \vdots \\ P_{1\to n} & P_{2\to n} & \cdots & P_{n\to n} \end{bmatrix}$$

=1 conservative system

time n, time n+1

$A\vec{x}[n] = \vec{x}[n+1]$

$A^{-1}\vec{x}[n] = \vec{x}[n-1]$

if the inverse exists

↳ note inverse is UNIQUE

Calculating Matrix Inv.

$[A \mid I_n] \to ge \to [I_n \mid A^{-1}]$

note: A^{-1} doesn't have to be in RREF

note: for $A = BC$, $m\times n$, $n\times m$: A^{-1} DNE because C has more columns than rows → LD

Basis

- for $\{v_1, \ldots, v_n\} = \mathcal{B}$, the vectors in $\mathcal{B}$ are a basis for $\mathbb{V}$ if (1) they're LI (2) their span is $\mathbb{V}$
- minimal set of spanning vectors
- for $\mathbb{R}^N$, N LI vectors form a basis

Dimension

- dimension($\mathbb{V}$) equals # vectors in its basis
- $\dim(\mathbb{R}^N) = N$

Subspace

$\mathbb{U}$ is a subspace of $\mathbb{V}$ if: (1) contains $\vec{0}$ (2) closed under vector + (3) closed under scalar ×

subspace basis = LI vectors spanning $\mathbb{U}$

subspace dimension = # vectors in basis

Columnspace

Col(A) where $A_{m\times n}$ = span n columns of A = range(A) — form a subspace for $\mathbb{R}^m$

Rowspace

= span n rows of A

Rank

= dim(col(A)) = # pivots in RREF

= dim(range(A)) — can be at most min(m,n) if A an n×m matrix

= dim(span(A)) — # columns

Rank-Nullity Thm

dim(range(A)) + dim(N(A)) = n (A matrix m×n, # cols)

Nullspace

- set of all $\vec{x}$ s.t. $A\vec{x} = \vec{0}$
- if $\vec{x} = \vec{0}$ is only soln, trivial nullspace
- solve for free vars, write as a vector, add

ex:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4, x_5 free vars

$x_2 = \alpha$, $x_5 = \gamma$, $x_4 = \beta$

$x_1 = -\alpha + 2\beta - 3\gamma$

$x_3 = \beta - \gamma$

$x_5 = \gamma$

write as vector sum

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\alpha + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\beta + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\gamma$$

$$N(A) = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

dim(N(A)) = 3

Eigenstuff

$A\vec{x} = \lambda\vec{x}$

- to calculate λs, solve $\det(A - \lambda I) = 0$ for λ
- to calculate $\vec{v}$s solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$(A - \lambda I)\vec{x} = 0$

1. Find λs — for an n×n matrix, we should have n λ's
2. Find eigenvectors corresponding by plugging in $\lambda_1, \ldots, \lambda_n$ into $(A - \lambda I)$

- if a matrix has m eigenvalues, all eigenvectors are LI
- if λ = 0 → not invertible, nontrivial nullspace
- λ = 1 → steady state
- for a 2×2 matrix: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$
- basis for N(A) = eigenvectors
- distinct eigenvectors form a subspace
- repeated e-values can have 1 or 2 e-vecs

steady-state $\vec{x}^* = P\vec{x}^*$

- to find steady state, substitute λ = 1, solve for nullspace and that's the st-state

ex: $(A - \lambda I)\vec{x}^* = 0$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 & 1/2 \\ 1/2 & -1 & 0 & 0 \\ 1/3 & 1/2 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \ (\lambda = 1) \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2/3 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x_1 = 2x_4 \\ x_2 = 2/3\, x_4 \\ x_3 = 3/2\, x_4 \end{array} \to \vec{x}^* = \begin{bmatrix} 2 \\ 2/3 \\ 3/2 \\ 1 \end{bmatrix} x_4$$

If you ~~[illegible]~~ start pumps with A_0, B_0 and C_0, what's the associated steady-state?

1. $A_0 + B_0 + C_0 = D$
2. given $x^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\alpha$, $x_1 + x_2 + x_3 = E$
3. $D = E\alpha$. Solve for α.
4. Multiply x^* by α to get ss for A_0, B_0 & C_0

Predicting system behavior for ~~[illegible]~~ initial states

$A^n\vec{x} = \alpha(\lambda^n\vec{x})$

- $\lambda > 1$: $\vec{x}[n] \to \infty$ exponential growth
- $\lambda = 1$: $\vec{x}[n] \to k\vec{v}$ (steady-state)
- $|\lambda| < 1$: $\vec{x}[n] \to \vec{0}$ exponential decay
- $\lambda = 1$: $\vec{x}[n] = 0\vec{v} \to \vec{0}$ instant. disappearance

$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ & $\vec{x}[n] = \alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \cdots + \alpha_n\lambda_n^n\vec{v}_n$

1. given an initial state, solve for $\alpha_1, \ldots, \alpha_n$: $\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{x}[0] \end{array}\right]$
2. Plug $\alpha_1, \ldots, \alpha_n$ into $\vec{x}[n]$ eqn and lambdas and $\vec{v}_i$

Rotation Matrix

$$A_R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Change of Basis

$\vec{r}^{(v)} = V^{-1}U\vec{r}^{(u)}$ $\vec{r}^{(v)} = V^{-1}\vec{r}$ $T = V^{-1}U$ ← transformation matrix

Diagonalization

- matrix $T_{n\times n}$ is diagonalizable iff it has n LI e-vectors with corresponding e-vals $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$; $\lambda_1, \ldots, \lambda_n$

$A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$ D = diagonal matrix of e-vals

$T = ADA^{-1}$ $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$

$T^k = AD^kA^{-1}$

Procedure
1. compute $(\vec{v}, \lambda)$ pairs of T
2. make sure e-vecs are LI (can use GE and nullspace to make sure it's trivial)
3. make A and A^{-1} using e-vecs
4. make D out of $\lambda_1 \ldots \lambda_n$
5. multiply and get T^k

Equivalent Statements — LI for $A_{n\times n}$ matrix

- $A\vec{x} = \vec{0}$ only when $\vec{x} = 0$
- $A^2 \neq 0$

- A has n pivot positions
- columns/rows form a basis for $\mathbb{R}^n$
- spans $\mathbb{R}^n$
- doesn't have an e-val of 0
- rank(A) = n
- A invertible
- A has trivial nullspace
- A has LI columns
- A is full rank
- det(A) ≠ 0
- $A\vec{x} = \vec{b}$ has a unique soln
- col(A) = $\mathbb{R}^n$

Thm: Span $\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\} = \mathbb{R}^2$

Known: span$\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\}$ set of all $\vec{b}$ that can be written as $\vec{b} = \alpha[\begin{smallmatrix}1\\1\end{smallmatrix}] + \beta[\begin{smallmatrix}1\\-1\end{smallmatrix}]$, $\alpha, \beta \in \mathbb{R}$

To show: span$\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\} = \mathbb{R}^2$ — any $\vec{b}$ in $\mathbb{R}^2$ can be represented using the span, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Want: All $\vec{b}$ to belong to the set S

$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right]$

$\to ge \to \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1 + b_2}{2} \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array}\right] \to \alpha = \frac{b_1 + b_2}{2}$ $\beta = \frac{b_1 - b_2}{2}$ ⇒ every $\vec{b} \in \mathbb{R}$ can be represented as a linear combo of α, β ⇒ $\vec{b} \in S$ ∎

Thm: If col(A) are LD, then $A\vec{x} = \vec{b}$ doesn't have a unique soln

Known: $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$

$\vec{a}_k = c_1 a_1 + c_2 a_2 + \cdots + c_{k-1}a_{k-1} + c_{k+1}a_{k+1} + \cdots + c_n a_n$

Want: $A\vec{x} = b$ doesn't have unique soln → either 0 or ∞. If possible, let a soln be $\vec{s}$.

rewrite as matrix: $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ 0 \\ \vdots \\ c_n \end{bmatrix} \to A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} - \begin{bmatrix} \vdots \end{bmatrix}\right) = \vec{0} \to A\begin{bmatrix} c_1 \\ \vdots \\ -1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$ call the vector $\vec{\omega}$: $A\vec{\omega} = 0$

(k-th position)

we know $A\vec{s} = \vec{b} \to A\vec{s} + \vec{0} = \vec{b} \to A\vec{s} + A\vec{\omega} = \vec{b} \to A(\vec{s} + \vec{\omega}) = \vec{b}$ $\vec{s} \neq \vec{\omega} \Rightarrow \vec{s} + \vec{\omega}$ also a soln!

→ $\vec{s}$ is not a unique soln ∎

Thm: $A\vec{x} = \vec{b}$ has 2+ solns then the cols of A are LD

Known: $A\vec{x} = \vec{b}$ has 2 distinct solutions $\vec{s}$, $\vec{\omega}$ but both solns

Want: columns of A are LD: $\vec{a}_k = c_1 a_1 + \cdots + c_n a_n$

$A\vec{s} = \vec{b}$, $A\vec{\omega} = b \to A\vec{\omega} = A\vec{s}$

$A(\vec{\omega} - \vec{s}) = \vec{0}$

Let $\vec{\omega} - \vec{s} = \vec{y} \to \vec{y} \neq \vec{0} \to \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \to A\vec{y} = \vec{0}$

$\to \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{0} \to y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n = \vec{0}$

At least one $y \neq 0$. call this y_k.

$y_k\vec{a}_k = y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n$

$\vec{a}_k = \frac{y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n}{y_k} \Rightarrow \vec{a}_k$ is lin. combo of other cols

Thm: If col(A) are LD, A is not invertible

Known: col(A) are LD: $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = 0$

Want: A^{-1} DNE. Assume A^{-1} does exist

Let A^{-1} exist. $\to A^{-1}A = AA^{-1} = I \to [a_1\ a_2 \cdots a_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$

$\to A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0 \to A^{-1}A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A^{-1}\vec{0} \to I\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \to \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$

contradiction! → A^{-1} can't exist ∎

Thm: If A is invertible, unique soln exists

Known: $A^{-1}A = AA^{-1} = I$

To show: (1) at least 1 soln (2) only 1 soln (every other soln is equal)

Guess: $\vec{s} = A^{-1}\vec{b}$ is a soln

Check: $A\vec{s} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b} \Rightarrow \vec{s}$ is a soln

Say $\vec{\omega} \neq \vec{s}$ is also a soln: $A\vec{\omega} = \vec{b} \to A^{-1}A\vec{\omega} = A^{-1}\vec{b} \Rightarrow \vec{\omega} = A^{-1}\vec{b}$, $\vec{\omega} = \vec{s}$

If Matrix A is invertible, its col are LI

A^{-1} exists → $A\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$

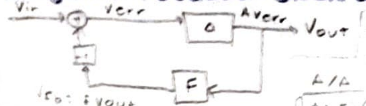
say $A\vec{v} = \vec{0} \to A^{-1}A\vec{v} = A^{-1}\vec{0} \to \vec{v} = \vec{0}$ ∎

Tips for proofs

- unique soln? proof by contradiction
- want columns of A? matrix mult
- if $A\vec{x} = \vec{0}$ then you can add this to pretty much anything

$C_1 \parallel C_2$
$= \frac{C_1 C_2}{C_1 + C_2}$

Negative Feedback Block diagram



Ex: $A \to$ very large

$V_{fb} = F V_{out}$
$V_{out} = A V_{err}$
$V_{err} = V_{in} - F V_{out}$
$V_{out} = A(V_{in} - F V_{out})$
$V_{out} = \frac{A}{1+AF} V_{in}$

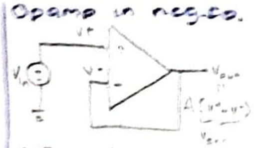
$\frac{A/A}{(1+AF)/A} = \frac{1}{1/A + F}$

$\lim_{A\to\infty} \frac{1}{\frac{1}{A}+F} = \frac{1}{F}$

$V_{out} = \frac{1}{F} V_{in}$

$V_{in} = F V_{out}$
$V_{fb} = F V_{out}$

Opamp in neg fb.



* If $V_{out}\uparrow$ and $V_{in}\downarrow$, we are in neg fb.
$V_- = V_{out}$ $V_+ = V_{in}$
$V_{out} = A(V_+ - V_-)$
$V_{out} = A V_{in} - A V_{out}$
$V_{out} = [A/(1+A)] V_{in}$
$V_{out} = \frac{1}{\frac{1}{A}+1} V_{in}$

as $A \to \infty$ $V_{out} = V_{in}$
↳ this is neg fb with $F = 1$

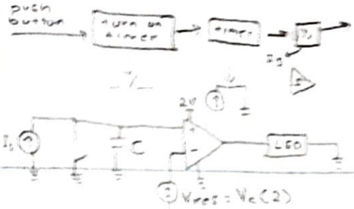
"wiggle"
↳ if $V_{out}\uparrow$, then $V_-\uparrow$
~~because~~ V_- is subtracted from V_+, then $(V_+ - V_-)\downarrow$
↳ this pulls $V_{out}\downarrow$, cancelling the increase

Restated:
if $V_{out}\uparrow$ then $(V_+ - V_-)\downarrow$ in neg fb
so $V_{out} = A(V_+ - V_-)\downarrow$

Design-Timer

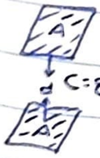
goal: circuit that, 2 secs after a button is pushed, it applies 2V to an LED.

Block



Note: Before button is pushed, S is closed to short circuit and make sure there is no current across C.
- gives us $V_c(0) = 0$
- $I_c(t) = 0$
(path of less resistance than wire of c)

After button pushed:
$V_c(t) = \frac{I_s t}{C}$, $V_{ref} = V_c(2) = \frac{I_s 2}{C}$



$C = \varepsilon\frac{A}{d}$

Find the power dissapated by the voltage source.

$P = VI = -\frac{V^2}{R} = -\frac{V^2}{6R}$

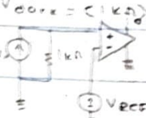
Derive an exp for Ctank.

$C_{tank} = C_{air} + C_{H_2O}$
$C_{air} = \frac{\varepsilon(h_{tot} - h_{H_2O})w}{w}$
$= \varepsilon h_{tot} - h_{H_2O}$
$C_{H_2O} = 81\varepsilon h_{H_2O}$
$C_{tank} = \varepsilon h_{tot} + 80 h_{H_2O}$

Circuit that outputs 10V w/light 0V w/o.

given:
$I_{light,dark}$
I_{light} 5mA I_{dark} = 1mA
① one op amp
② one resistor
③ two voltage sources

$V_{light} = (1k\Omega)(5mA) = 5V$
$V_{dark} = (1k\Omega)(1mA) = 1V$



- ~~circuit~~ light sensor can only provide max of 40mW. Is this a problem?
$P = IV = (5mA)(-5V) = -25mW$
- No, not a problem bc our circuit can generate 25mW which is less than P_{max}.
- Resistances that meet the
- P_{max} req: $|P| \le 40mW \Rightarrow$
$|I^2 V| \le 40mW \Rightarrow |(5mA)^2 R| \le 40mW$
$R \le 1.6k\Omega$

What to use as a refr. voltage?

Halfway between the 2 options.
$V_{ref} = \frac{V^+ + V^-}{2}$

Design a motor driving circuit that outputs a decreasing positive motor voltage as the robot moves toward the light.

Spec:
- Distance ↓, speed ↓
- $V_m \ge 5V$ when far $(D\uparrow)$
- "far away": $R_{PH} = 10k\Omega$
"close by": $R_{PH} = 100\Omega$

Given
op amps, 10V, -10V, resistors

Strategy
- decr. as a func of distance
$D\uparrow$ ↳↓ $R_{PH}\uparrow$

Question
- Can we build something to measure resistance?
↳ output voltage
A/ let's try a voltage divider
V-DIV
$V_x = \frac{R_2}{R_1+R_2}(10V) = \frac{1}{1+\frac{R_1}{R_2}}$
$R_1\uparrow$ $(\frac{R_1}{R_2})\uparrow$ $V_x\downarrow$
$R_2\uparrow$ $(\frac{R_1}{R_2})\downarrow$ $V_x\uparrow$ } make $R_2 = R_{PH}$

Now we have
$D\uparrow$ L↓ $R_{PH}\uparrow$ $(\frac{R_1}{R_{PH}})\downarrow$ $V_x\uparrow$

From spec
$V_x = 5V$ when $R_{PH} = 10k\Omega$
↳ plug R_{PH} into V_x eqn and solve for R_1.
$\Rightarrow R_1 = 10k\Omega$

Connect!
10V, 10kΩ, R_{PH}, R_m
Need buffer to avoid loading.

Design a comparator circuit that outputs a positive motor voltage when robot exceeds 1m in distance, making the robot move toward it, and a negative voltage when robot is w/in 1m (making robot move away)

Specs
① "stop" at 1m/1kΩ
- output > 0 when >1m $(\uparrow D)$
- output < 0 when <1m $(\downarrow D)$
② Recall $R_{PH}\uparrow$, D↓ and $R_{PH}\downarrow$ when D↑

Given
resistors, comparator, 10V, -10V

Design
$V_x = \frac{R_{PH} V_{in}}{R_{ref} + R_{PH}}$
$D\uparrow$, $R_{PH}\uparrow$ $V_x\uparrow$

Pick $R_{ref} = 1k\Omega \Rightarrow V_x = 5V$ at 1m

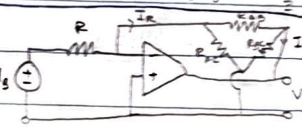
10V, 1kΩ, R_{PH}, V_{out}, -10V
5V → make 5V? → 10V, 1kΩ, 1kΩ

Finally
10V, 1kΩ, R_{PH}, 10V, -10V, R_m, 1kΩ, 1kΩ

"gain"
usually voltage gain
$G = \frac{V_{out}}{V_{in}}$ $V_{out} = G(V_+ - V_-)$

$I = \frac{dQ}{dt}$

charge on a cap after time t
$Q = It$



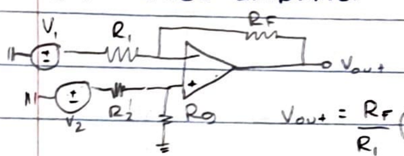
$V_{out,1} = ?$
$I_R = \frac{V_B}{R} = I_{RAB}$
$V_{out,1} = -R_{AB} I_{RAB} = -R_{AB} I_R$
$= -\frac{R_{AB}}{R} V_B$

max s.t.:
$-1.5mV \le V_{out,1} \le -.5mV$
i thru electrodes is $< 100\mu A$
$V_B = 5V$

① Pick R.
$V_{out,1} = -.5mV$ to $-1.5mV$
$= -\frac{50\Omega}{R} 5V$ to $-\frac{150\Omega}{R} 5V$
$R = 500k\Omega$

② $I_{RAB,max} = \frac{V_{out,1}}{R_{AB}} = \frac{V_B}{R}$
$= \frac{5V}{500k\Omega} = 10\mu A$
$I_{RBC,max} = \frac{V_{out,1}}{R_{BC}}$
$= \frac{V_B R_{AB}}{R \cdot R_{BC}}$
I_{RBC} will be largest when R_{AB} is largest and R_{BC} is smallest
$I_{RBC,max} = \frac{150\Omega \cdot 5V}{50\Omega \cdot 500k\Omega} = 30\mu A$

Difference amplifier



$V_{out} = \frac{R_F}{R_1}(V_2 - V_1)$

$\frac{ab}{a+b} \parallel c = \frac{\frac{ab}{a+b}c}{\frac{ab}{a+b}+c}$

Properties

Scalar multiplication

distributive over vector add

Orthogonality

$\vec{x}, \vec{y}$ are orthog if $\langle \vec{x}, \vec{y} \rangle = 0$

Normalization

unit vector in same direction

$\frac{\vec{x}}{\|\vec{x}\|}$

Inner product and θ

$$\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|} = \cos\theta$$

$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$

Norm (Euclidean)

$$\|\vec{x}\| = \sqrt{(x_1)^2 + \cdots + x_n^2} = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

Properties

1) non-negativity: $\|\vec{x}\| > 0$ (for $\vec{x} \neq \vec{0}$)
2) $\vec{0}$ is only vector with a norm of 0: $\|\vec{x}\| = 0$ iff $\vec{x} = 0$
3) scalar multiplication: $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$
4) triangle inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Cauchy-Schwarz

$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$

note $|\cos\theta| \leq 1$ for all real θ

$$|\langle \vec{x}, \vec{y} \rangle| = \big|\|\vec{x}\|\|\vec{y}\|\cos\theta\big| = \|\vec{x}\|\|\vec{y}\| \cdot |\cos\theta| \leq \|\vec{x}\|\|\vec{y}\|$$

Vector Operations - Addition

$$\vec{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\langle \vec{x}, \vec{y} \rangle = x_0 + x_1 + \cdots + x_n$ (adds all terms in $\vec{x}$)

Projection (2D)

$$\text{proj}_{\vec{y}}\,\vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}\vec{y}$$

projection of $\vec{x}$ onto $\vec{y}$

1D Projection (Scalar)

(Scalar projection of $\vec{b}$ onto $\vec{a}$)

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle}\vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}\vec{a}$$

Vector Operations - Averaging

$$\vec{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$$

$$\langle \vec{x}, \vec{y} \rangle = \frac{x_0 + x_1 + \cdots + x_n}{n}$$

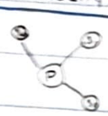
Distance to satellite

τ = time delay

$d = \tau v$ (light / sound)

distance = (time delay)(velocity)

Trilateration



1)
$$\|\vec{p} - \vec{s}_1\|^2 = d_1^2$$
$$\|\vec{p} - \vec{s}_2\|^2 = d_2^2$$
$$\|\vec{p} - \vec{s}_3\|^2 = d_3^2$$

2)
$$\|\vec{p}\|^2 - 2\langle \vec{s}_1, \vec{p} \rangle + \|\vec{s}_1\|^2 = d_1^2$$
$$\|\vec{p}\|^2 - 2\langle \vec{s}_2, \vec{p} \rangle + \|\vec{s}_2\|^2 = d_2^2$$
$$\|\vec{p}\|^2 - 2\langle \vec{s}_3, \vec{p} \rangle + \|\vec{s}_3\|^2 = d_3^2$$

3)
$$2(\vec{s}_1 - \vec{s}_2)^T\vec{p} = \|\vec{s}_1\|^2 - \|\vec{s}_2\|^2 - d_1^2 + d_2^2$$
$$2(\vec{s}_1 - \vec{s}_3)^T\vec{p} = \|\vec{s}_1\|^2 - \|\vec{s}_3\|^2 - d_1^2 + d_3^2$$

4)
$$\begin{bmatrix} 2(\vec{s}_1 - \vec{s}_2)^T \\ 2(\vec{s}_1 - \vec{s}_3)^T \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \|\vec{s}_1\|^2 - \|\vec{s}_2\|^2 - d_1^2 + d_2^2 \\ \|\vec{s}_1\|^2 - \|\vec{s}_3\|^2 - d_1^2 + d_3^2 \end{bmatrix}$$

Note: ~~Assuming ideal~~ conditions (no satellites are colinear), we need n satellites to solve for n unknowns. One dimension is always lost bc we need to use 1 eqn to linearize the remaining equations

Circular Cross Correlation

$$\text{circorr}(\vec{r}, \vec{s}) = \begin{bmatrix} \langle \vec{r}, \vec{s}^{(0)} \rangle \\ \langle \vec{r}, \vec{s}^{(1)} \rangle \\ \langle \vec{r}, \vec{s}^{(2)} \rangle \end{bmatrix} = \begin{bmatrix} - \vec{s}^{(0)T} - \\ - \vec{s}^{(1)T} - \\ - \vec{s}^{(2)T} - \end{bmatrix}\begin{bmatrix} | \\ \vec{r} \\ | \end{bmatrix}$$

Find τ using corr

$$\hat{\tau} = \text{argmax}\,(\text{circorr}(\vec{r}, \vec{s})[k])$$

Cross correlation

correlation of $\vec{r}$ with $\vec{s}$

$$\text{corr}_{\vec{r}}(\vec{s})[k] = \sum_{i=-\infty}^{\infty} r(i)s(i-k)$$

↑ at element k

$\text{corr}_{\vec{r}}\vec{s} = [\_\ \_\ \_\ \_ \ldots \_]$

↑ the largest value is in the same position corresponding to the shift.

Say max value is 5, occuring at position 2, then the shift is 2

$\text{corr}_{\vec{r}}(\vec{s})[k] = \langle \vec{r}, \vec{s}^{(d)} \rangle$

Properties

$\text{corr}_{\vec{x}}(\vec{y}) \neq \text{corr}_{\vec{y}}(\vec{x})$

$\text{circorr}(\vec{x}, \vec{y}) \neq \text{circorr}(\vec{y}, \vec{x})$

Proof(ish) for 2D projection (Least squares)

$A\vec{x} + \vec{e} = \vec{b}$

want closest vector to $\vec{b}$ in the span of the columns of A

$(x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m) + \vec{e} = \vec{b}$ $\text{span}\{\vec{a}_i\}$

For 2D:

$x\vec{a} + \vec{e} = \vec{b}$

$\|\vec{e}\| = \|\vec{b} - x\vec{a}\|$

Thm: $\vec{e}$ is orthogonal to every vector in the columnspace of A iff it is orthogonal to each of the columns $\vec{a}_i$ that form the basis of its columnspace

$\vec{v} \in \text{span}(A)$

$\vec{v} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_m\vec{a}_m$

$\langle \vec{e}, \vec{a}_i \rangle = 0$

$$\langle \vec{e}, \vec{v} \rangle = \langle \vec{e}, \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_m\vec{a}_m \rangle$$
$$= \langle \vec{e}, \alpha_1\vec{a}_1 \rangle + \langle \vec{e}, \alpha_2\vec{a}_2 \rangle + \cdots + \langle \vec{e}, \alpha_m\vec{a}_m \rangle$$
$$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \cdots + \alpha_m \cdot 0$$
$$= 0$$

$\langle \vec{e}, \vec{a}_i \rangle = \vec{a}_i^T\vec{e} = 0$

$$\begin{bmatrix} -\vec{a}_1^T- \\ -\vec{a}_2^T- \\ \vdots \\ -\vec{a}_n^T- \end{bmatrix}\vec{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow A^T\vec{e} = 0$$

Least squares

$$\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$$
$$\hat{\vec{b}} = A(A^TA)^{-1}A^T\vec{b}$$

Least squares derivation

$A^T(\vec{b} - A\hat{\vec{x}}) = \vec{0}$

$A^TA\hat{\vec{x}} = A^T\vec{b}$

$\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$

$\text{Null}(A^TA) = \text{Null}(A)$

Least Squares and Orbits

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y = 1$$

$$\underbrace{\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\vec{v} = (A^TA)^{-1}A^T\vec{b}$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix} = (A^TA)^{-1}A^T\vec{b}$$

Computing distances from time delays

If beacon i is a distance d_i from the receiver:

$t_i = \frac{d_i}{v}$ $d_i = vt_i$

Unsynchronized clocks (TDOA)

Instead of starting recording at $t = 0$, what if our received signal started at $t = \Delta t$

then, we should use

$t_i = \frac{d_i}{v} - \Delta t$

$$t_0 = \frac{d_0}{v} - \Delta t$$
$$t_1 = \frac{d_1}{v} - \Delta t$$
$$\vdots$$
$$t_{m-1} = \frac{d_{m-1}}{v} - \Delta t$$

$\xrightarrow{\text{subtract } t_0}$

$$d_1 - d_0 = v(t_1 - t_0)$$
$$d_2 - d_0 = v(t_2 - t_0)$$
$$\vdots$$
$$d_{m-1} - d_0 = v(t_{m-1} - t_0)$$

TDOA: τ

$\tau_i = t_i - t_0$

choose origin to be position of 0th beacon and put receiver at position $\vec{x}$ in the coordinate system

let coordinates of ith beacon be $\vec{a}_i$ with $\vec{a}_0 = \vec{0}$

$d_0^2 = \vec{x}^T\vec{x}$

$\sqrt{(\vec{x} - \vec{a}_i)^T(\vec{x} - \vec{a}_i)} - \sqrt{\vec{x}^T\vec{x}} = v\tau_i$

↳ for all $\vec{a}_i$

↓ square

$(\vec{x} - \vec{a}_i)^T(\vec{x} - \vec{a}_i) = v^2\tau_i^2 + \vec{x}^T\vec{x} + 2v\tau_i\sqrt{\vec{x}^T\vec{x}}$

↳ for all $\vec{a}_i$

linearize & subtract

$$\begin{bmatrix} \frac{2\vec{a}_1^T}{v\tau_1} - \frac{2\vec{a}_2^T}{v\tau_2} \\ \frac{2\vec{a}_1^T}{v\tau_1} - \frac{2\vec{a}_3^T}{v\tau_3} \\ \vdots \\ \frac{2\vec{a}_1^T}{v\tau_1} - \frac{2\vec{a}_{m-1}^T}{v\tau_{m-1}} \end{bmatrix}\vec{x} = \begin{bmatrix} v\tau_2 - v\tau_1 + \frac{\vec{a}_1^T\vec{a}_1}{v\tau_1} - \frac{\vec{a}_2^T\vec{a}_2}{v\tau_2} \\ v\tau_3 - v\tau_1 + \frac{\vec{a}_1^T\vec{a}_1}{v\tau_1} - \frac{\vec{a}_3^T\vec{a}_3}{v\tau_3} \\ \vdots \\ v\tau_{m-1} - v\tau_1 + \frac{\vec{a}_1^T\vec{a}_1}{v\tau_1} - \frac{\vec{a}_{m-1}^T\vec{a}_{m-1}}{v\tau_{m-1}} \end{bmatrix}$$

$\vec{x} \in \mathbb{R}^d \rightarrow$ need $n \geq d+2$ beacons to uniquely determine location product | orthogonality | Norm ... | Cauchy-Schwarz

Given 2 eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two unique eigenvalues λ_1 and λ_2 of a 2×2 matrix A, $\vec{v}_1, \vec{v}_2$ form a basis

Known
$A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \neq \vec{v}_2 \neq \vec{0}$
$A \in \mathbb{R}^{2\times2}$

Want
$\vec{v}_1, \vec{v}_2$ form a basis for $\mathbb{R}^2$
① $\vec{v}_1, \vec{v}_2$ are LI ② $\vec{v}_1, \vec{v}_2$ span all of $\mathbb{R}^2$

If possible, let $\vec{v}_1$; $\vec{v}_2$ be LD
$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$ → say $\alpha_1 \neq 0$ → $\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\vec{v}_2$
Multiply by A.
$A\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}A\vec{v}_2$
$A\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$
$\lambda_1\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$

→ plug in $\vec{v}_1$ into the eqn.
$-\frac{\alpha_2}{\alpha_1}\lambda_1\vec{v}_2 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$
$\lambda_1 = \lambda_2$
↳ contradiction!

→ therefore $\vec{v}_1, \vec{v}_2$ are LI
To show they span all of $\mathbb{R}^2$:
$[\vec{v}_1\ \vec{v}_2 \mid \vec{x}]$ → $V = [\vec{v}_1\ \vec{v}_2]$
→ V is an invertible matrix
⇒ $[V \mid \vec{x}]$ has a unique soln

→ therefore $\vec{x} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$
⇒ $\{\vec{v}_1, \vec{v}_2\}$ form a basis for $\mathbb{R}^2$

If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ are LD vectors in in $\mathbb{R}^n$, then $\{A\vec{v}_1, A\vec{v}_2, ..., A\vec{v}_n\}$ are LD.

Known:
$\vec{v}_i = \sum_{j\neq i} \alpha_j\vec{v}_j$

Show:
$A\vec{v}_i = \sum_{l\neq i} \beta_l\vec{v}_l$

$A\vec{v}_i = A\left(\sum_{j\neq i}\alpha_j\vec{v}_j\right)$
$= \sum_{j\neq i} A(\alpha_j\vec{v}_j)$
$A\vec{v}_i = \sum_{j\neq i}\alpha_j(A\vec{v}_j)$ ⇒ linear combo exists
QED

If $\vec{v}_1, \vec{v}_2$ solns to $A\vec{x} = \vec{b}$, $\vec{v}_1 + \vec{v}_2$ ✓, $\vec{b}$ must be $\vec{0}$.
Known:
$A\vec{v}_1 = \vec{b}$, $A\vec{v}_2 = \vec{b}$
Want: $\vec{b} = \vec{0}$
$A(\vec{v}_1 + \vec{v}_2) = \vec{b}$ → $A\vec{v}_1 + A\vec{v}_2 = \vec{b}$
⇒ $\vec{b} + \vec{b} = \vec{b}$ ⇒ $\vec{b} = \vec{0}$
QED

$\text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n\}$
say $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$
$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$
$= \alpha_1(\vec{v}_1 + \vec{v}_2) + (-\alpha_1 + \alpha_2)\vec{v}_2 + ... + \alpha_n\vec{v}_n$
⇒ $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n\}$

say $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n\}$
$\vec{r} = \beta_1(\vec{v}_1 + \vec{v}_2) + \beta_2(\vec{v}_2) + ... + \beta_n\vec{v}_n$
$= \beta_1\vec{v}_1 + (\beta_1 + \beta_2)\vec{v}_2 + ... + \beta_n\vec{v}_n$
$\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$

If A invertible, unique A^{-1}
Known
$AA^{-1} = A^{-1}A = I$
Want
A^{-1} unique
say B_1, B_2 inverses of A, $B_1 \neq B_2$
$AB_1 = B_1A = I$ $AB_2 = B_2A = I$
$B_2AB_1 = B_2B_1A$
$(B_2A)B_1 = B_2(B_1A)$
$B_1 = B_2$
→ contradiction, A^{-1} must be unique

If $QP = I$ and $RQ = I$, then $P = R$.
$QP = RQ$
$RQP = RRQ$
$\underbrace{(RQ)}_{I}P = R\underbrace{(RQ)}_{I}$
$P = R$

Transpose
- eigenvalues remain the same across transposes

If a system of K reservoirs has columns that sum to one, then if s is the total amount of water at timestep n, then the total amount of water is s at timestep n+1

Known
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
$a_{11} + a_{21} = 1$
$a_{12} + a_{22} = 1$
$x_1[n] + x_2[n] = s$
$\vec{x}[n+1] = A\vec{x}[n]$
Consider product
$A\vec{x}[n] = \vec{x}[n+1]$

mat mul
$\begin{bmatrix} a_{11}x_1[n] + a_{12}x_2[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$

want
$x_1[n+1] + x_2[n+1] = s$

Matrix Inverse Properties
- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $(KA)^{-1} = K^{-1}A^{-1}$ $K \in \mathbb{R}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Applying Matrices
- go from right to left. ex.

$ABCD\vec{x}$
$(A(B(C(D\vec{x}))))$
④ ③ ② ①

Given unknown Matrix A, given $A\vec{v}_1 = A\vec{v}_2 = \vec{p}$, find $\vec{u}$ s.t. $A\vec{u} = \vec{0}$ where $\vec{u} \neq \vec{0}$.
$A\vec{v}_1 - A\vec{v}_2 = \vec{0}$
$A(\underbrace{\vec{v}_1 - \vec{v}_2}) = \vec{0}$
$\vec{u} = \vec{v}_1 - \vec{v}_2$

Let $U = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$. Calculate $U^T \cdot U$ for orthogonalized vectors in U
$U^TU = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_3^T \end{bmatrix}[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$
since $\vec{u}_i^T\vec{u}_j$ (the off-diagonal terms) are orthonogalized, they equal 0.
⇒ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}$

$\lambda_i\vec{v}_i = A\vec{v}_i$
⇒ $A^{-1}\lambda_i\vec{v}_i = \vec{v}_i$
$A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$

More steady-state:
$\vec{s}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$
To decompose $\vec{s}[0]$ into the eqn (given $\vec{s}[0]$):
$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \vec{s}[0]$
↳ do GE

- ~~$A\vec{s}[0]$ has lambdas ($A\vec{s}[0] = \lambda\vec{s}[0]$)~~

Is $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ in $C(A)$ when $a = 3$?

$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ $a = 3$ $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are LI ? span $\mathbb{R}^2$ → yes

Solve for smallest possible identical e-vals for A.
$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ → $\det(A - \lambda I) = 0$ → $\lambda^2 - (2+a)\lambda + (2a+1) = 0$
→ $\lambda = \frac{2+a \pm \sqrt{(2+a)^2 - 4(2a+1)}}{2}$ → since we want identical evals, everything under sqrt = 0.
→ $(2+a)^2 - 4(2a+1) = 0$ → solve for a → a = 0, 4
→ want a minimizing e-vals → plug in a to the quadratic eqn formula w/ the λs → (a = 4 → λ = 3);
(a = 0 → λ = 1) → $\boxed{a = 0}$

Find all vals for x s.t. A has a trivial nullspace
$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix}$ → GE → $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2x \end{bmatrix}$ → want LI so $x \neq 0$

Given a transformation, what is the transformation matrix that created the transform?
ex: $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix} \to \begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$

rewrite:
$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$

→ do mat mul, solve for a, b, c, d and plug into A
$a = \frac{2}{3}$ $b = 0$ $c = 0$ $d = \frac{2}{3}$
⇒ $A = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$

4b

Pluto — ac